

Blow-up results for semilinear wave equations in the super-conformal case

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Abstract

We consider the semilinear wave equation in higher dimensions with power nonlinearity in the super-conformal range, and its perturbations with lower order terms, including the Klein-Gordon equation. We improve the upper bounds on blow-up solutions previously obtained by Killip, Stovall and Viřan [6]. Our proof uses the similarity variables' setting. We consider the equation in that setting as a perturbation of the conformal case, and we handle the extra terms thanks to the ideas we already developed in [5] for perturbations of the pure power case with lower order terms.

Keywords: Semilinear wave equation, finite time blow-up, blow-up rate, super-conformal exponent.

AMS classification : 35L05, 35L67, 35B20.

1 Introduction

This paper is devoted to the study of blow-up solutions for the following semilinear wave equation:

$$\begin{cases} \partial_t^2 u = \Delta u + |u|^{p-1}u + f(u) + g(x, t, \nabla u, \partial_t u), \\ (u(x, 0), \partial_t u(x, 0)) = (u_0(x), u_1(x)) \in H_{loc}^1(\mathbb{R}^N) \times L_{loc}^2(\mathbb{R}^N), \end{cases} \quad (1.1)$$

in spatial dimensions $N \geq 2$, where $u(t) : x \in \mathbb{R}^N \rightarrow u(x, t) \in \mathbb{R}$ and $p_c < p < p_S$, where $p_c \equiv 1 + \frac{4}{N-1}$ is the conformal critical exponent and $p_S \equiv 1 + \frac{4}{N-2}$ is the

Sobolev critical exponent. Moreover, we take $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^{2N+2} \rightarrow \mathbb{R}$ \mathcal{C}^1 functions satisfying

$$\begin{aligned} (H_f) \quad & |f(u)| \leq M(1 + |u|^q), \quad \text{for all } u \in \mathbb{R} \quad \text{with } (q < p, \ M > 0), \\ (H_g) \quad & |g(x, t, v, z)| \leq M(1 + |v| + |z|), \quad \text{for all } x, v \in \mathbb{R}^N, t, z \in \mathbb{R} \quad \text{with } (M > 0). \end{aligned}$$

We would like to mention that equation (1.1) encompasses the case of the following nonlinear Klein-Gordon equation

$$\partial_t^2 u = \Delta u + |u|^{p-1}u - u, \quad (x, t) \in \mathbb{R}^N \times [0, T]. \quad (1.2)$$

In order to keep our analysis clear, we only give the proof for the following non perturbed equation

$$\partial_t^2 u = \Delta u + |u|^{p-1}u, \quad (x, t) \in \mathbb{R}^N \times [0, T], \quad (1.3)$$

and refer the reader to [4] and [5] for straightforward adaptations to equation (1.1).

The Cauchy problem of equation (1.3) is solved in $H_{loc}^1 \times L_{loc}^2$. This follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$, valid whenever $1 < p < p_S$. The existence of blow-up solutions for the associated ordinary differential equation of (1.3) is a classical result. By using the finite speed of propagation, we conclude that there exists a blow-up solution $u(t)$ of (1.3) which depends non trivially on the space variable. In this paper, we consider a blow-up solution $u(t)$ of (1.3), we define (see for example Alinhac [1] and [2]) Γ as the graph of a function $x \mapsto T(x)$ such that the domain of definition of u is given by

$$D_u = \{(x, t) \mid t < T(x)\}.$$

The set D_u is called the maximal influence domain of u . Moreover, from the finite speed of propagation, T is a 1-Lipschitz function. The graph Γ is called the blow-up graph of u .

Let us first introduce the following non-degeneracy condition for Γ . If we introduce for all $x \in \mathbb{R}^N$, $t \leq T(x)$ and $\delta > 0$, the cone

$$C_{x,t,\delta} = \{(\xi, \tau) \neq (x, t) \mid 0 \leq \tau \leq t - \delta|\xi - x|\}, \quad (1.4)$$

then our non degeneracy condition is the following: x_0 is a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0, 1) \text{ such that } u \text{ is defined on } C_{x_0, T(x_0), \delta_0}. \quad (1.5)$$

We aim at studying the growth estimate of $u(t)$ near the space-time blow-up graph in the super-conformal case (where $p_c < p < p_S$).

Let us briefly mention some results concerning the blow-up rate of solutions of semilinear wave equations. The first result valid for general solutions is due to

Merle and Zaag in [8] (see also [7] and [9]) who proved, that if $1 < p \leq p_c$ and u is a solution of (1.3), then the growth estimate near the space-time blow-up graph is given by the associated ODE. In [4] and [5], we extend the result of Merle and Zaag to perturbed equations of type (1.1) under some reasonable growth estimates on f and g in (1.1) (see hypothesis (H_f) and (H_g)). Note that, in all these papers, the method crucially relies on the existence of a Lyapunov functional in similarity variables established by Antonini and Merle [3]. Recently, Killip, Stovall and Viřan in [6] have shown, among other results, that the results of Merle and Zaag remain valid for the semilinear Klein-Gordon equation (1.2). Moreover, they consider also the case where $p_c < p < p_S$ and prove that, if u is a solution of (1.2), then for all $x_0 \in \mathbb{R}^N$, there exists $K > 0$ such that, for all $t \in [0, T(x_0))$,

$$(T(x_0) - t)^{-\frac{(p-1)N}{p+3}} \int_{B(x_0, \frac{T(x_0)-t}{2})} u^2(x, t) dx \leq K, \quad (1.6)$$

and for all $t \in (0, T(x_0)]$,

$$\int_{T(x_0)-t}^{T(x_0)-\frac{t}{2}} \int_{B(x_0, \frac{T(x_0)-\tau}{2})} (|\nabla u(x, \tau)|^2 + |\partial_t u(x, \tau)|^2) dx d\tau \leq K. \quad (1.7)$$

Moreover, if x_0 is a non characteristic point, then they use a covering argument to obtain the same estimates with the ball $B(x_0, \frac{T(x_0)-\tau}{2})$ replaced by the ball $B(x_0, T(x_0) - \tau)$ in the inequalities (1.6) and (1.7).

Here, we obtain a better result thanks to a different method based on the use of self-similar variables. This method allows us to improve the results of [6] as we state in the following:

THEOREM 1 (*Growth estimate near the blow-up surface for Eq. (1.1)*). *If u is a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$, then for all $x_0 \in \mathbb{R}^N$ and $t \in [0, T(x_0))$, we have*

$$(T(x_0) - t)^{-\frac{(p-1)N}{p+3}} \int_{B(x_0, T(x_0)-t)} u^2(x, t) dx \rightarrow 0, \quad \text{as } t \rightarrow T(x_0). \quad (1.8)$$

Moreover, for all $t \in (0, T(x_0)]$, we have

$$\int_{T(x_0)-t}^{T(x_0)-\frac{t}{2}} \int_{B(x_0, \frac{T(x_0)-\tau}{2})} |\partial_t u(x, \tau)|^2 dx d\tau \leq K_1, \quad (1.9)$$

and

$$\int_{T(x_0)-t}^{T(x_0)-\frac{t}{2}} \int_{B(x_0, \frac{T(x_0)-\tau}{2})} |\nabla u(x, \tau)|^2 dx d\tau \leq K_1. \quad (1.10)$$

If in addition x_0 is a non characteristic point, then we have for all $t \in (0, T(x_0)]$,

$$\int_{T(x_0)-t}^{T(x_0)-\frac{t}{2}} \int_{B(x_0, T(x_0)-\tau)} \left(|\nabla u(x, \tau)|^2 + |\partial_t u(x, \tau)|^2 \right) dx d\tau \rightarrow 0, \quad \text{as } t \rightarrow 0. \quad (1.11)$$

Moreover, we have

$$\begin{aligned} & \frac{T(x_0) - t}{2} \int_{B(x_0, T(x_0)-t)} \left(|\nabla u(x, t)|^2 - \left(\frac{x - x_0}{T(x_0) - t} \cdot \nabla u(x, t) \right)^2 \right. \\ & \left. + |\partial_t u(x, t)|^2 - \frac{1}{p+1} |u(x, t)|^{p+1} \right) dx \rightarrow 0, \quad \text{as } t \rightarrow T(x_0). \end{aligned} \quad (1.12)$$

REMARK 1.1 *i) Let us remark that, we have the following lower bound which follows from standard techniques (scaling arguments, the wellposedness in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, the finite speed of propagation and the fact that x_0 is a non characteristic point): there exist $\varepsilon_0 > 0$, such that*

$$\begin{aligned} 0 < \varepsilon_0 & \leq (T(x_0) - t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} \\ & + (T(x_0) - t)^{\frac{2}{p-1}+1} \left(\frac{\|\partial_t u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} \right). \end{aligned}$$

ii) In Theorem 1, we improve recent results of Killip, Stovall and Viřan in [6]. More precisely, we obtain a better estimate in (1.8) and if x_0 is non characteristic point we have the better estimate (1.11).

iii) Up to a time dependent factor, the expression in (1.12) is equal to the main terms of the energy in similarity variables (see (1.20)). However, even with this improvement, we think that our estimates are still not optimal.

iv) The constant K_1 , and the rate of convergence to 0 of the different quantities in the previous theorem and in the whole paper, depend only on N , p and the upper bound on $T(x_0)$, $1/T(x_0)$, and the initial data (u_0, u_1) in $H^1(B(x_0, 2T(x_0))) \times L^2(B(x_0, 2T(x_0)))$, together with $\delta_0(x_0)$ if x_0 is non characteristic point.

Our method relies on the estimates in similarity variables introduced in [3] and used in [7], [8] and [9]. More precisely, given (x_0, T_0) such that $0 < T_0 \leq T(x_0)$, we introduce the following self-similar change of variables:

$$y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t), \quad u(x, t) = \frac{1}{(T_0 - t)^{\frac{2}{p-1}}} w_{x_0, T_0}(y, s). \quad (1.13)$$

This change of variables transforms the backward light cone with vortex (x_0, T_0) into the infinite cylinder $(y, s) \in B \times [-\log T_0, +\infty)$. In the new set of variables (y, s) , the behavior of u as $t \rightarrow T_0$ is equivalent to the behavior of w as $s \rightarrow +\infty$.

From (1.3), the function w_{x_0, T_0} (we write w for simplicity) satisfies the following equation for all $y \in B \equiv B(0, 1)$ and $s \geq -\log T_0$:

$$\begin{aligned} \partial_s^2 w + \frac{p+3}{p-1} \partial_s w + 2y \cdot \nabla \partial_s w &= \sum_{i,j} (\delta_{i,j} - y_i y_j) \partial_{y_i y_j}^2 w - \frac{2p+2}{(p-1)^2} y \cdot \nabla w \\ &\quad - \frac{2p+2}{(p-1)^2} w + |w|^{p-1} w. \end{aligned} \quad (1.14)$$

Putting this equation in the following form

$$\begin{aligned} \partial_s^2 w &= \operatorname{div}(\nabla w - (y \cdot \nabla w)y) + 2\eta y \cdot \nabla w - \frac{2p+2}{(p-1)^2} w + |w|^{p-1} w \\ &\quad - \frac{p+3}{p-1} \partial_s w - 2y \cdot \nabla \partial_s w, \quad \forall y \in B \text{ and } s \geq -\log T_0, \end{aligned} \quad (1.15)$$

where

$$\eta = \frac{N-1}{2} - \frac{2}{p-1} = \frac{2}{p_c-1} - \frac{2}{p-1} > 0, \quad (1.16)$$

the key idea of our paper is to view this equation as a perturbation of the conformal case (corresponding to $\eta = 0$) already treated in [5] with the term $2\eta y \cdot \nabla w$. Of course, this term is not a lower order term with respect to the nonlinearity. For that reason, we will have exponential growth rates in the w setting. Let us emphasize the fact that our analysis is not just a trivial adpatation of our previous work [5].

The equation (1.14) will be studied in the Hilbert space \mathcal{H}

$$\mathcal{H} = \left\{ (w_1, w_2), \left| \int_B \left(w_2^2 + |\nabla w_1|^2 (1 - |y|^2) + w_1^2 \right) dy < +\infty \right. \right\}.$$

In the conformal case where $p = p_c$, Merle and Zaag [8] proved that

$$E_0(w) = \int_B \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} |\nabla w|^2 - \frac{1}{2} (y \cdot \nabla w)^2 + \frac{p+1}{(p-1)^2} w^2 - \frac{|w|^{p+1}}{p+1} \right) dy, \quad (1.17)$$

is a Lyapunov functional for equation (1.14). When $p > p_c$, we introduce

$$E(w) = E_0(w) + I(w), \quad (1.18)$$

where

$$I(w) = -\eta \int_B w \partial_s w dy + \frac{\eta N}{2} \int_B w^2 dy, \quad (1.19)$$

and η is defined in (1.16). Finally, we define the energy function as

$$F(w, s) = E(w) e^{-2\eta s}. \quad (1.20)$$

The proof of Theorem 1 crucially relies on the fact that $F(w, s)$ is a Lyapunov functional for equation (1.14) on the one hand, and on the other hand, on a blow-up criterion involving $F(w, s)$. Indeed, with the functional $F(w, s)$ and some more work, we are able to adapt the analysis performed in [8]. In the following, we show that $F(w, s)$ is a Lyapunov functional:

PROPOSITION 1.2 (*Existence of a decreasing functional for Eq. (1.14)*).

For all $s_2 > s_1 \geq -\log T_0 = s_0$, the functional $F(w, s)$ defined in (1.20) satisfies

$$\begin{aligned} F(w(s_2), s_2) - F(w(s_1), s_1) &= - \int_{s_1}^{s_2} e^{-2\eta s} \int_{\partial B} (\partial_s w - \eta w)^2 d\sigma ds \\ &\quad - \frac{\eta(p-1)}{p+1} \int_{s_1}^{s_2} e^{-2\eta s} \int_B |w|^{p+1} dy ds. \end{aligned} \quad (1.21)$$

Moreover, for all $s \geq s_0$, we have $F(w, s) \geq 0$.

This paper is organized as follows: In section 2, we prove Proposition 1.2. Using this result, we prove Theorem 1 in section 3.

2 Existence of a decreasing functional for equation (1.14) and a blow-up criterion

Consider u a solution of (1.3) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$, and consider its self-similar transformation w_{x_0, T_0} defined at some scaling point (x_0, T_0) by (1.13) where $T_0 \leq T(x_0)$. This section is devoted to the proof of Proposition 1.2. We proceed in two parts:

- In subsection 2.1, we show the existence of a decreasing functional for equation (1.14).
- In subsection 2.2, we prove a blow-up criterion involving this functional.

2.1 Existence of a decreasing functional for equation (1.14)

In this subsection, we prove that the functional $F(w, s)$ defined in (1.20) is decreasing. More precisely we prove that the functional $F(w, s)$ satisfies the inequality (1.21). Now we state two lemmas which are crucial for the proof. We begin with bounding the time derivative of $E_0(w)$ defined in (1.17) in the following lemma.

LEMMA 2.1 For all $s \geq -\log T_0$, we have

$$\frac{d}{ds}(E_0(w)) = - \int_{\partial B} (\partial_s w)^2 d\sigma + 2\eta \int_B (\partial_s w)^2 dy + 2\eta \int_B \partial_s w (y \cdot \nabla w) dy. \quad (2.1)$$

Proof. Multiplying (1.15) by $\partial_s w$ and integrating over the ball B , we obtain for all $s \geq -\log T_0$,

$$\frac{d}{ds}(E_0(w)) = -2 \int_B \partial_s w (y \cdot \nabla \partial_s w) dy - \frac{p+3}{p-1} \int_B (\partial_s w)^2 dy + 2\eta \int_B \partial_s w (y \cdot \nabla w) dy.$$

Since we see from integration by parts that

$$-2 \int_B \partial_s w (y \cdot \nabla \partial_s w) dy = - \int_B y \cdot \nabla (\partial_s w)^2 dy = N \int_B (\partial_s w)^2 dy - \int_{\partial B} (\partial_s w)^2 d\sigma,$$

this concludes the proof of Lemma 2.1. \blacksquare

We are now going to prove the following estimate for the functional $I(w)$:

LEMMA 2.2 *For all $s \geq -\log T_0$, we have*

$$\begin{aligned} \frac{d}{ds} I(w) &= 2\eta E(w) - 2\eta \int_B (\partial_s w)^2 dy - \frac{\eta(p-1)}{p+1} \int_B |w|^{p+1} dy \\ &\quad - 2\eta \int_B \partial_s w (y \cdot \nabla w) dy - \eta^2 \int_{\partial B} w^2 d\sigma + 2\eta \int_{\partial B} w \partial_s w d\sigma. \end{aligned} \quad (2.2)$$

Proof: Note that $I(w)$ is a differentiable function for all $s \geq -\log T_0$ and that

$$\frac{d}{ds} I(w) = -\eta \int_B (\partial_s w)^2 dy - \eta \int_B w \partial_s^2 w dy + \eta N \int_B w \partial_s w dy.$$

By using equation (1.15) and integrating by parts, we have

$$\begin{aligned} \frac{d}{ds} I(w) &= -\eta \int_B (\partial_s w)^2 dy + \eta \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) dy - \eta \int_B |w|^{p+1} dy \\ &\quad - 2\eta^2 \int_B w (y \cdot \nabla w) dy + \eta \frac{2p+2}{(p-1)^2} \int_B w^2 dy \\ &\quad + 2\eta \int_B w (y \cdot \nabla \partial_s w) dy + \eta \left(\frac{p+3}{p-1} + N \right) \int_B w \partial_s w dy. \end{aligned}$$

Then by integrating by parts, we have

$$\begin{aligned} \frac{d}{ds} I(w) &= -\eta \int_B (\partial_s w)^2 dy + \eta \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) dy - \eta \int_B |w|^{p+1} dy \\ &\quad - \eta^2 \int_{\partial B} w^2 d\sigma + \left(\eta^2 N + \eta \frac{2p+2}{(p-1)^2} \right) \int_B w^2 dy \\ &\quad + 2\eta \int_{\partial B} w \partial_s w d\sigma - 2\eta \int_B (y \cdot \nabla w) \partial_s w dy + \eta \left(\frac{p+3}{p-1} - N \right) \int_B w \partial_s w dy. \end{aligned} \quad (2.3)$$

By combining (1.17), (1.18), (1.19), (1.16) and (2.3), we conclude the proof of Lemma 2.2. \blacksquare

From Lemmas 2.1 and 2.2, we are in a position to prove the first part of Proposition 1.2.

Proof of the first part of Proposition 1.2: From Lemmas 2.1 and 2.2, we obtain for all $s \geq -\log T_0$,

$$\frac{d}{ds} E(w) = 2\eta E(w) - \int_{\partial B} \left(\partial_s w - \eta w \right)^2 d\sigma - \frac{\eta(p-1)}{p+1} \int_B |w|^{p+1} dy.$$

Therefore, using the definition of the functional $F(w, s)$ in (1.20), we write

$$\frac{d}{ds}F(w, s) = -e^{-2\eta s} \int_{\partial B} \left(\partial_s w - \eta w \right)^2 d\sigma - \frac{\eta(p-1)}{p+1} e^{-2\eta s} \int_B |w|^{p+1} dy. \quad (2.4)$$

By integration, we get (1.21). This concludes the first part of the proof of Proposition 1.2. \blacksquare

2.2 A blow-up criterion

We finish the proof of Proposition 1.2 here. More precisely, for all $x_0 \in \mathbb{R}^N$ and $T_0 \in (0, T(x_0)]$, we prove that

$$\forall s \geq -\log T_0, \quad F(w_{x_0, T_0}(s), s) \geq 0. \quad (2.5)$$

We give the proof only in the case where x_0 is a non characteristic point. Note that the case where x_0 is a characteristic point can be done exactly as in Appendix A page 119 in [10].

Proof of the last point of Proposition 1.2: The argument is the same as in the corresponding part in [3]. We write the proof for completeness. Arguing by contradiction, we assume that there exists a non characteristic point $x_0 \in \mathbb{R}^N$, $T_0 \in (0, T(x_0)]$ and $s_1 \geq -\log T_0$ such that $F(w(s_1), s_1) < 0$, where $w = w_{x_0, T_0}$. Since the energy $F(w(s), s)$ decreases in time, we have $F(w(1+s_1), 1+s_1) < 0$. Consider now for $\delta > 0$ the function $\tilde{w}^\delta(y, s) = w_{x_0, T_0-\delta}(y, s)$. From (1.13), we see that for all $(y, s) \in B \times [1+s_1, +\infty)$

$$\tilde{w}^\delta(y, s) = \frac{1}{(1 + \delta e^s)^{\frac{2}{p-1}}} w\left(\frac{y}{1 + \delta e^s}, -\log(\delta + e^{-s})\right).$$

- (A) Note that \tilde{w}^δ is defined in $B \times [1+s_1, +\infty)$, whenever $\delta > 0$ is small enough such that $-\log(\delta + e^{-1-s_1}) \geq s_1$.
- (B) By construction, \tilde{w}^δ is also a solution of equation (1.14).
- (C) For δ small enough, we have $F(\tilde{w}^\delta(1+s_1), 1+s_1) < 0$ by continuity of the function $\delta \mapsto F(\tilde{w}^\delta(1+s_1), 1+s_1)$.

Now, we fix $\delta = \delta_0 > 0$ such that (A), (B) and (C) hold. Since $F(\tilde{w}^{\delta_0}, s)$ is decreasing in time, we have

$$\liminf_{s \rightarrow +\infty} F(\tilde{w}^{\delta_0}(s), s) \leq F(\tilde{w}^{\delta_0}(1+s_1), 1+s_1) < 0. \quad (2.6)$$

Let us note that we have

$$-\eta \int_B \tilde{w}^{\delta_0} \partial_s \tilde{w}^{\delta_0} dy \geq -\frac{1}{2} \int_B (\partial_s \tilde{w}^{\delta_0})^2 dy - \frac{\eta^2}{2} \int_B (\tilde{w}^{\delta_0})^2 dy \quad (2.7)$$

By (1.17), (1.18), (1.19), (2.7) and the fact that $\eta \in [0, 1]$, we deduce

$$\begin{aligned} E(\tilde{w}^{\delta_0}(s)) &\geq \left(\frac{\eta N}{2} - \frac{\eta^2}{2}\right) \int_B (\tilde{w}^{\delta_0})^2 dy - \frac{1}{p+1} \int_B |\tilde{w}^{\delta_0}|^{p+1} dy \\ &\geq -\frac{1}{p+1} \int_B |\tilde{w}^{\delta_0}|^{p+1} dy. \end{aligned} \quad (2.8)$$

So, by (1.20), we have

$$F(\tilde{w}^{\delta_0}(s), s) \geq -\frac{e^{-2\eta s}}{p+1} \int_B |\tilde{w}^{\delta_0}|^{p+1} dy. \quad (2.9)$$

After a change of variables, we find that

$$F(\tilde{w}^{\delta_0}(s), s) \geq -\frac{e^{-2\eta s}}{(p+1)(1+\delta_0 e^s)^{\frac{4}{p-1}+2-N}} \int_B |w(z, -\log(\delta_0 + e^{-s}))|^{p+1} dz.$$

Since we have $-\log(\delta_0 + e^{-s}) \rightarrow -\log \delta_0$ as $s \rightarrow +\infty$ and since $\|w(s)\|_{L^{p+1}(B)}$ is locally bounded from the fact that $w = w_{x_0, T_0}$ and x_0 is non characteristic point, by a continuity argument, it follows that the former integral remains bounded and

$$F(\tilde{w}^{\delta_0}(s), s) \geq -\frac{C e^{-2\eta s}}{(1+\delta_0 e^s)^{\frac{4}{p-1}+2-N}} \rightarrow 0, \quad (2.10)$$

as $s \rightarrow +\infty$ (use the fact that $\frac{4}{p-1} + 2 - N - 2\eta = 1$ and $\eta > 0$). So, from (2.10), it follows that

$$\liminf_{s \rightarrow +\infty} F(\tilde{w}^{\delta_0}(s), s) \geq 0. \quad (2.11)$$

From (2.6), this is a contradiction. Thus (2.5) holds. This concludes the proof of Proposition 1.2. \blacksquare

3 Proof of Theorem 1

Consider u a solution of (1.3) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$. Translating Theorem 1 in the self-similar setting w_{x_0, T_0} (we write w for simplicity) defined by (1.13), our goal becomes the following Proposition:

PROPOSITION 3.1 *If u is a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$, then for all $x_0 \in \mathbb{R}^N$ and $T_0 \leq T(x_0)$, we have for all $s \geq s_0 = -\log T_0$,*

$$e^{-2\eta s} \int_s^{s+1} \int_B \left((\partial_s w(y, \tau))^2 + |\nabla w(y, \tau)|^2 (1 - |y|^2) \right) dy d\tau \leq K. \quad (3.1)$$

Moreover,

$$e^{-2\eta s} \int_B |w(y, s)|^{\frac{p+3}{2}} dy \rightarrow 0, \quad \text{as } s \rightarrow +\infty, \quad (3.2)$$

$$e^{\frac{-8\eta s}{p+3}} \int_B |w(y, s)|^2 dy \rightarrow 0, \quad \text{as } s \rightarrow +\infty. \quad (3.3)$$

If in addition x_0 is a non characteristic point, then we have,

$$e^{-2\eta s} \int_s^{s+1} \int_B \left((\partial_s w(y, \tau))^2 \right) dy d\tau \rightarrow 0, \quad (3.4)$$

$$e^{-2\eta s} \int_s^{s+1} \int_B \left(|\nabla w(y, \tau)|^2 \right) dy d\tau \rightarrow 0, \quad (3.5)$$

as $s \rightarrow +\infty$. Moreover, we have

$$F(w, s) \rightarrow 0, \quad \text{as } s \rightarrow +\infty. \quad (3.6)$$

In this section, we prove Proposition 3.1 which directly implies Theorem 1, as in the proof of Theorem 1.1, (page 1145) in [8].

Let us first use Proposition 1.2 and the averaging technique of [9] and [8] to get the following bounds:

LEMMA 3.2 *For all $s \geq s_0 = -\log T_0$, we have*

$$0 \leq F(w(s), s) \leq F(w(s_0), s_0), \quad (3.7)$$

$$\int_{s_0}^{\infty} e^{-2\eta s} \int_B |w(y, s)|^{p+1} dy ds \leq \frac{p+1}{\eta(p-1)} F(w(s_0), s_0), \quad (3.8)$$

$$\int_{s_0}^{\infty} e^{-2\eta s} \int_{\partial B} \left(\partial_s w(\sigma, s) - \eta w(\sigma, s) \right)^2 d\sigma ds \leq F(w(s_0), s_0). \quad (3.9)$$

If in addition x_0 is non characteristic (with a slope $\delta_0 \in (0, 1)$), then

$$e^{-2\eta s} \int_s^{s+1} \int_B \left(\partial_s w_{x_0, T_0}(y, \tau) - \lambda(\tau, s) w_{x_0, T_0}(y, \tau) \right)^2 dy d\tau \rightarrow 0, \quad \text{as } s \rightarrow +\infty, \quad (3.10)$$

where $0 \leq \lambda(\tau, s) \leq C(\delta_0)$, for all $\tau \in [s, s+1]$.

Proof: The first three estimates are a direct consequence of Proposition 1.2. As for the last estimate, by introducing $f(y, s) = e^{-\eta s} w(y, s)$, we see that the dispersion estimate (3.9) can be written as follows:

$$\int_{s_0}^{\infty} \int_{\partial B} \left(\partial_s f(\sigma, s) \right)^2 d\sigma ds \leq F(w(s_0), s_0). \quad (3.11)$$

In particular, we have

$$\int_s^{s+1} \int_{\partial B} \left(\partial_s f(\sigma, \tau) \right)^2 d\sigma d\tau \rightarrow 0, \quad \text{as } s \rightarrow +\infty. \quad (3.12)$$

By exploiting (3.12) where the space integration is done over the unit sphere, one can use the averaging technique of Proposition 4.2 (page 1147) in [8] to get the same estimate with the space variable integrated over the whole ball B . \blacksquare

From Lemma 3.2, we are in a position to prove Proposition 3.1

Proof of Proposition 3.1:

- *Proof of (3.1):* By integrating the functional $F(w, s)$ defined in (1.20) in time between s and $s + 1$, we obtain:

$$\begin{aligned} & \int_s^{s+1} e^{-2\eta\tau} \int_B \left((\partial_s w)^2 + |\nabla w|^2 (1 - |y|^2) \right) dy d\tau = -\frac{2(p+1)}{(p-1)^2} \int_s^{s+1} e^{-2\eta\tau} \int_B w^2 dy d\tau \\ & 2 \int_s^{s+1} F(w(\tau), \tau) d\tau - \int_s^{s+1} e^{-2\eta\tau} \int_B \left(|\nabla w|^2 |y|^2 - (y \cdot \nabla w)^2 \right) dy d\tau \\ & + \frac{2}{p+1} \int_s^{s+1} e^{-2\eta\tau} \int_B |w|^{p+1} dy d\tau + \underbrace{2\eta \int_s^{s+1} e^{-2\eta\tau} \int_B \left(w \partial_s w - \frac{N}{2} w^2 \right) dy d\tau}_{A(s)}. \end{aligned} \quad (3.13)$$

Now, we control all the terms on the right-hand side of the relation (3.13):

Note that the first term is negative, while the second term is bounded because of the bound (3.7) on the energy $F(w, s)$. Since $|y \cdot \nabla w| \leq |y| |\nabla w|$, we can say that the third is also negative. Remark that (3.8) implies that the fourth term is also bounded. Finally, it remains only to control the term $A(s)$.

Combining the Cauchy-Schwarz inequality, the inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, and the fact that $N \geq 2$ and $\eta \in [0, 1]$, we write

$$A(s) \leq \frac{1}{2} \int_s^{s+1} e^{-2\eta\tau} \int_B (\partial_s w)^2 dy d\tau. \quad (3.14)$$

Now, we are able to conclude the proof of the inequality (3.1). For this, we combine (3.13), (3.14) and the above-mentioned arguments for the first four terms to get

$$\int_s^{s+1} e^{-2\eta\tau} \int_B \left((\partial_s w)^2 + |\nabla w|^2 (1 - |y|^2) \right) dy d\tau \leq K + \frac{1}{2} \int_s^{s+1} e^{-2\eta\tau} \int_B (\partial_s w)^2 dy d\tau. \quad (3.15)$$

The desired bound in (3.1) follows then from (3.15).

- *Proof of (3.2):* Using the mean value theorem, we derive the existence of $\sigma(s) \in [s, s+1]$ such that

$$\int_B |w(y, \sigma(s))|^{\frac{p+3}{2}} dy = \int_s^{s+1} \int_B |w(y, \tau)|^{\frac{p+3}{2}} dy d\tau. \quad (3.16)$$

By Jensen's inequality, we have

$$\int_s^{s+1} \int_B |w(y, \tau)|^{\frac{p+3}{2}} dy d\tau \leq C \left(\int_s^{s+1} \int_B |w(y, \tau)|^{p+1} dy d\tau \right)^{\frac{p+3}{2(p+1)}}. \quad (3.17)$$

By combining (3.16) and (3.17), we can write that

$$\int_B |w(y, \sigma(s))|^{\frac{p+3}{2}} dy \leq C \left(\int_s^{s+1} \int_B |w(y, \tau)|^{p+1} dy d\tau \right)^{\frac{p+3}{2(p+1)}}. \quad (3.18)$$

Using (3.18) and the fact that $ab \leq a^2 + b^2$, we have

$$\begin{aligned} \int_B |w(y, s)|^{\frac{p+3}{2}} dy &\leq \int_B |w(y, \sigma(s))|^{\frac{p+3}{2}} dy + C \int_s^{s+1} \int_B |\partial_s w(y, \tau)| |w(y, \tau)|^{\frac{p+1}{2}} dy d\tau \\ &\leq C \left(\int_s^{s+1} \int_B |w(y, \tau)|^{p+1} dy d\tau \right)^{\frac{p+3}{2(p+1)}} \\ &\quad + C \left(\int_s^{s+1} \int_B |w(y, \tau)|^{p+1} dy d\tau \right)^{\frac{1}{2}} \left(\int_s^{s+1} \int_B |\partial_s w(y, \tau)|^2 dy d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Since $e^{-2\eta s} \int_s^{s+1} \int_B |w(y, \tau)|^{p+1} dy d\tau \rightarrow 0$ from (3.8), we use (3.1) to obtain (3.2).

- *Proof of (3.3)*: It follows from (3.2) through the Holder inequality and (3.2).

- *Proof of (3.4)*: Note that from now on, we assume that x_0 is a non characteristic point with slope $\delta_0 \in (0, 1)$. It is a direct consequence of (3.3) and (3.10).

- *Proof of (3.5)*: Let $s \geq s_0 + 1$, $s_1 = s_1(s) \in [s - 1, s]$ and $s_2 = s_2(s) \in [s, s + 1]$ to be chosen later. By integrating after multiplication by $e^{-2\eta s}$ the expression (2.3) of $I(w)$ in time between s_1 and s_2 , we obtain

$$\begin{aligned} \eta \int_{s_1}^{s_2} e^{-2\eta s} \int_B |\nabla w|^2 (1 - |y|^2) dy ds &= \underbrace{e^{-2\eta s_2} I(w(s_2)) - e^{-2\eta s_1} I(w(s_1))}_{B_1(s)} \\ &+ \underbrace{2\eta \int_{s_1}^{s_2} e^{-2\eta s} \int_B (y \cdot \nabla w) \partial_s w dy ds}_{B_2(s)} + \underbrace{\eta \int_{s_1}^{s_2} e^{-2\eta s} \int_B (\partial_s w)^2 dy ds}_{B_3(s)} \\ &+ \underbrace{\eta \int_{s_1}^{s_2} e^{-2\eta s} \int_B |w|^{p+1} dy ds}_{B_4(s)} - \underbrace{(\eta^2 N + \eta \frac{2p+2}{(p-1)^2}) \int_{s_1}^{s_2} e^{-2\eta s} \int_B w^2 dy ds}_{B_5(s)} \\ &- \underbrace{\eta \left(\frac{p+3}{p-1} - N \right) \int_{s_1}^{s_2} e^{-2\eta s} \int_B w \partial_s w dy ds}_{B_6(s)} - \underbrace{\int_{s_1}^{s_2} e^{-2\eta s} \int_{\partial B} (\partial_s w)^2 d\sigma ds}_{B_7(s)} \\ &+ \underbrace{\int_{s_1}^{s_2} e^{-2\eta s} \int_{\partial B} (\partial_s w - \eta w)^2 d\sigma ds}_{B_8(s)} - \underbrace{\eta \int_{s_1}^{s_2} e^{-2\eta s} \int_B (|y|^2 |\nabla w|^2 - (y \cdot \nabla w)^2) dy ds}_{B_9(s)}. \end{aligned} \quad (3.19)$$

Now, we control all the terms on the right-hand side of the relation (3.19):

Note that, by (1.19) and using the Cauchy-Schwarz inequality, we can write

$$e^{-2\eta s_2} I(w(s_2)) \leq C e^{-2\eta s_2} \int_B (\partial_s w(s_2))^2 dy + C e^{-2\eta s_2} \int_B w^2(s_2) dy. \quad (3.20)$$

By exploiting (3.3) and the fact that $s_2 \in [s, s+1]$, we conclude that

$$e^{-2\eta s_2} \int_B w^2(s_2) dy \rightarrow 0 \quad \text{as } s \rightarrow +\infty, \quad (3.21)$$

on the one hand. On the other hand, by using the mean value theorem, let us choose $s_2 = s_2(s) \in [s, s+1]$ such that

$$\int_s^{s+1} e^{-2\eta \tau} \int_B (\partial_s w(\tau))^2 dy d\tau = e^{-2\eta s_2} \int_B (\partial_s w(s_2))^2 dy. \quad (3.22)$$

By combining (3.4) and (3.22) we obtain

$$e^{-2\eta s_2} \int_B (\partial_s w(s_2))^2 dy \rightarrow 0 \quad \text{as } s \rightarrow +\infty. \quad (3.23)$$

Then, by using (3.20), (3.21) and (3.23), we get

$$e^{-2\eta s_2} I(w(s_2)) \rightarrow 0 \quad \text{as } s \rightarrow +\infty. \quad (3.24)$$

From (1.19) and the fact that $ab \leq a^2 + b^2$, we write

$$-e^{-2\eta s_1} I(w(s_1)) \leq C e^{-2\eta s_1} \int_B (\partial_s w(s_1))^2 dy. \quad (3.25)$$

Similarly, by using the mean value theorem, we choose $s_1 = s_1(s) \in [s-1, s]$ such that

$$\int_{s-1}^s e^{-2\eta \tau} \int_B (\partial_s w(\tau))^2 dy d\tau = e^{-2\eta s_1} \int_B (\partial_s w(s_1))^2 dy. \quad (3.26)$$

By (3.4), (3.25) and (3.26) we obtain

$$-e^{-2\eta s_1} I(w(s_1)) \rightarrow 0 \quad \text{as } s \rightarrow +\infty. \quad (3.27)$$

Note that by combining (3.24), (3.27) and the fact that $B_1(s) = e^{-2\eta s_2} I(w(s_2)) - e^{-2\eta s_1} I(w(s_1))$, we deduce that

$$B_1(s) \rightarrow 0 \quad \text{as } s \rightarrow +\infty. \quad (3.28)$$

To estimate $B_2(s)$, since $s_1 \in [s-1, s]$ and $s_2 \in [s, s+1]$, we write

$$B_2(s) \leq C \left(\int_{s-1}^{s+1} e^{-2\eta \tau} \int_B |\nabla w|^2 dy d\tau \right)^{\frac{1}{2}} \left(\int_{s-1}^{s+1} e^{-2\eta \tau} \int_B (\partial_s w)^2 dy d\tau \right)^{\frac{1}{2}}. \quad (3.29)$$

By using (3.1) and the covering argument of [8], we have

$$e^{-2\eta s} \int_s^{s+1} \int_B |\nabla w|^2 dy d\tau \leq K. \quad (3.30)$$

Thus

$$\int_{s-1}^{s+1} e^{-2\eta\tau} \int_B |\nabla w|^2 dy d\tau \leq CK. \quad (3.31)$$

Then, by (3.29), (3.31) and (3.4), we deduce

$$B_2(s) \rightarrow 0 \text{ as } s \rightarrow +\infty. \quad (3.32)$$

By (3.4), we can say that

$$B_3(s) \rightarrow 0 \text{ as } s \rightarrow +\infty. \quad (3.33)$$

By (3.8), we also deduce that

$$B_4(s) \rightarrow 0 \text{ as } s \rightarrow +\infty. \quad (3.34)$$

The terms $B_5(s)$ and $B_7(s)$ are negative. By using (3.3) and (3.4), we have

$$B_6(s) \rightarrow 0 \text{ as } s \rightarrow +\infty. \quad (3.35)$$

By (3.9), we write that

$$B_8(s) \rightarrow 0 \text{ as } s \rightarrow +\infty. \quad (3.36)$$

Finally, since $|y \cdot \nabla w| \leq |y| |\nabla w|$, we can say that the term $B_9(s)$ is negative. By combining (3.28), (3.32), (3.33), (3.34), (3.35), (3.36) and the fact that the terms $B_5(s)$, $B_7(s)$ and $B_9(s)$ are negative, we conclude that

$$\int_s^{s+1} e^{-2\eta\tau} \int_B |\nabla w|^2 (1 - |y|^2) dy d\tau \rightarrow 0 \text{ as } s \rightarrow +\infty. \quad (3.37)$$

By using (3.37) and the covering argument of [8], we deduce that estimate (3.5) holds.

- *Proof of (3.6):* By integrating the functional $F(w, s)$ defined in (1.20) in time between s and $s + 1$, we write

$$\begin{aligned} \int_s^{s+1} F(w, \tau) d\tau &= \int_s^{s+1} \int_B e^{-2\eta\tau} \left(\frac{1}{2} (\partial_s w)^2 + \frac{p+1}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) dy d\tau \\ &\quad + \frac{1}{2} \int_s^{s+1} e^{-2\eta\tau} \int_B \left(|\nabla w|^2 - (y \cdot \nabla w)^2 \right) dy d\tau \\ &\quad - \eta \int_s^{s+1} e^{-2\eta\tau} \int_B w \partial_s w dy d\tau + \frac{\eta N}{2} \int_s^{s+1} e^{-2\eta\tau} \int_B w^2 dy d\tau. \end{aligned} \quad (3.38)$$

By using (3.3), (3.5), (3.8) and (3.38), we conclude that

$$\int_s^{s+1} F(w, \tau) d\tau \rightarrow 0, \quad \text{as } s \rightarrow +\infty. \quad (3.39)$$

Combining The monotonicity of $F(w, s)$ proved in Proposition 1.2, and (3.39), we deduce the identity (3.6). This concludes the proof of Proposition 3.1. ■

Since the derivation of Theorem 1 from Proposition 3.1 is the same as in [8] (up to some very minor changes), this concludes the proof of Theorem 1. ■

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